2.8. THE CAUCHY INTEGRAL AND RELATED INTEGRAL OPERATORS†

Let Γ be a rectifiable curve in $\mathbb{C}$. The Cauchy integral of a function defined on Γ and integrable relative to arc length is defined as

$$C(f)(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

Recently Calderón [1] has proved the existence almost everywhere of nontangential boundary values for the function C(f) [denoted as $C(f)(\zeta)$]. This pointwise existence theorem follows from the following estimate proved in [1] by an ingenious complex variable method.

THEOREM 1 (A. P. Calderón). There is a constant η_0, $\eta_0 > 0$, such that for all functions φ with $\|\varphi'\|_\infty<\eta_0$ there exists a constant C_φ for which

$$\int\left|\int_{-\infty}^{\infty}\frac{f(t)\,dt}{(s-t)+i(\varphi(s)-\varphi(t))}\right|^2 ds\leq C_\varphi\int|f(s)|^2ds.$$

It is not hard (by using singular integral techniques) to reduce the existence a.e. result mentioned above to Theorem 1. Several important questions remain open.

1. Is the restriction $\|\varphi'\|_\infty<\eta_0$ necessary to obtain the estimate of Theorem 1? Calderón's method as well as other techniques are unable to eliminate this restriction.

2. Since the operator $C(f)(\zeta)$ exists almost everywhere for all functions in $L^2(\Gamma, |d\zeta|)$, it is natural to conjecture the existence of a weight $\omega_\Gamma(\zeta)$ (>0 a.e.) for which

$$\int|C(f)(\zeta)|^2\omega_\Gamma(\zeta)|d\zeta|\leq C_\Gamma\int|f|^2|d\zeta|.$$

(The existence of such a weight for a weak L^2 estimate is guaranteed by general considerations related to the Nikishin–Stein theorem.)

3. The integral operator appearing in Theorem 1 is related to a general class of operators like Hilbert transforms, of which the following are typical examples.

a) The so-called commutators of order n

$$A_n(f)=\int_{-\infty}^{\infty}\left(\frac{A(x)-A(y)}{x-y}\right)^n\frac{f(y)}{x-y}\,dy;$$

b)
$$\int_0^\infty\frac{A(x+t)-2A(x)+A(x-t)}{t^2}f(x-t)\,dt;$$

c)
$$\int\frac{A(x)-A(y)}{|x-y|^{2+\gamma}}f(y)\,dy,\quad \gamma>0.\ \text{(Here } A'\in L^\infty.)$$

It is easily seen that Theorem 1 is equivalent to the following estimates on the operators A_n:

$$\|A_n(f)\|_2\leq c^n\|A'\|_\infty^n\|f\|_2 \tag{*}$$

for some constant c. The boundedness in L^2 of the operators in a), b), c) has been proved in [2, 3] by using Fourier analysis and real variable techniques (which extend to $\mathbb{R}^n$). Unfortunately the estimate obtained (by these methods) on the growth of the constant in (*) is

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of the order of n! (and not c^n). It will be highly desirable to obtain a proof of Calderón's result which does not depend on special tricks or complex variables. Any such technique will extend to higher dimensions and is bound to imply various sharp estimates for operators arising in partial differential equations.

LITERATURE CITED

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